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On Involutive Homogeneous Varieties and Representations of Weyl Algebras

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This paper is concerned with the geometry of minimal involutive homogeneous varieties in complex affine $2n$-space and its application to the study of the representation theory of the nth complex Weyl algebra A_n. The main results are the existence of minimal involutive homogeneous varieties of any given codimension, and of A_n-modules that have these varieties for characteristic variety. We also determine conditions on the codimensions of A_n-modules M and N under which $\mathrm{Ext}^1(M, N)$ is a finite dimensional vector space.

1. INTRODUCTION

The representation theory of the complex nth Weyl algebra A_n has been a much studied subject. A key problem is the description of the irreducible representations which, in this case, are so numerous that they seem to defy classification.

In fact, the irreducible modules over A_n are very poorly understood when $n \geq 2$. For instance, all the examples known up to 1985 had Gelfand–Kirillov dimension n. The first example of an irreducible module over A_n of dimension different from n was obtained by Stafford in [Sta]. Stafford wrote down an explicit operator of A_n and proved that it generated a maximal left ideal.

In 1989, Bernstein and Lunts introduced in [BL] a geometric method for the construction of irreducible A_n-modules of Gelfand–Kirillov dimension $2n - 1$. Their approach makes use of the most important geometric invariant of an A_n-module, its *characteristic variety*. This is a homogeneous

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subvariety of $\mathbf{C}^{2n}$, which is also involutive (co-isotropic) with respect to the standard symplectic structure of $\mathbf{C}^{2n}$. The key to the construction of irreducible modules is provided by the following definition. A homogeneous involutive subvariety of $\mathbf{C}^{2n}$ is *minimal* if it does not contain any proper involutive homogeneous subvarieties. All homogeneous involutive irreducible varieties of codimension n in $\mathbf{C}^{2n}$ are minimal. Examples in codimension 1 were obtained by Bernstein and Lunts. The importance of these varieties lies in the fact that, under a mild extra hypothesis, if a module over the Weyl algebra has a characteristic variety that is minimal involutive homogeneous, then it is an irreducible module; see [Cou3, Proposition 2.3(d)].

The purpose of this paper is the study of minimal homogeneous involutive varieties and their application to the representation theory of the Weyl algebra. In Sec. 3 we give a construction that yields examples of minimal involutive varieties of any desired codimension. Luckily the varieties constructed in these examples are clearly characteristic varieties. This allows us to show in Sec. 4 that there exist irreducible modules of all possible Gelfand–Kirillov dimensions over the nth Weyl algebra, whose characteristic varieties are minimal involutive homogeneous. We also use these techniques to determine when Ext^1 is a finite dimensional complex vector space.

In Sec. 5 we study Lagrangian homogeneous varieties. Of course such a variety can only be smooth if it is a linear subspace; otherwise the origin will be a singular point. The next best thing would be for the variety to be normal, which is the case we consider in some detail. Some problems suggested by the results of this paper are discussed in Sec. 6. Finally, Sec. 2 contains basic facts about symplectic geometry that are often used in the paper.

2. SYMPLECTIC GEOMETRY

Let ω_n be the standard symplectic form of $\mathbf{C}^{2n}$. We choose coordinates so that $\omega_n = \sum_{i=1}^{n} dx_{2i-1} \wedge dx_{2i}$. If L is a linear subspace of $\mathbf{C}^{2n}$, denote by $L^{\perp}$ its orthogonal space with respect to the form ω_n. If $L^{\perp} \subseteq L$, then L is said to be an *involutive*, or *co-isotropic*, subspace of $\mathbf{C}^{2n}$. We often use the following consequence of Darboux's theorem; see [Arn, p. 6].

THEOREM 2.1. *Let L be a linear involutive subspace of $\mathbf{C}^{2n}$ of codimension $k \leq n$. There exists a linear map τ in the complex symplectic group* $\mathrm{Sp}(2n)$ *such that $\tau(L)$ is defined by the equations $x_{2(n-k+1)} = x_{2(n-k+2)} = \cdots = x_{2n} = 0$.*

An algebraic variety X of $\mathbf{C}^{2n}$ is *involutive* if, at every non-singular point $p \in X$, the tangent space $T_pX \subseteq \mathbf{C}^{2n}$ is involutive. The non-degeneracy of ω_n implies that $\dim X \geq n$. The involutive varieties of dimension n are called *Lagrangian* varieties. They play a very important role in symplectic geometry; see [Arn, p. 35ff].

The involutivity condition can also be checked using the ideal of the variety. First note that the form ω_n gives rise to an isomorphism $\mathcal{I}\colon T^\star(\mathbb{C}^{2n}) \to T\mathbb{C}^{2n}$, of the cotangent into the tangent bundle. In other words, we have a correspondence between 1-forms and vector fields given by $\omega_n(\cdot, \mathcal{I}(\alpha)) = \alpha(\cdot)$. If $f, g \in S_n$, the *Poisson bracket* of f and g is defined by

$$\{f, g\} = \omega_n(\mathcal{I}\, df, \mathcal{I}\, dg).$$

Now a variety X of $\mathbb{C}^{2n}$ is *involutive* if and only if its ideal $I(X)$ is closed under the Poisson bracket; that is $\{I(X), I(X)\} \subseteq I(X)$. See [Cou1, Chap. 11, Sect. 2].

We will now describe the results from [BL] and [Lun] that are used in this paper. An algebraic variety $X \subseteq \mathbf{C}^{2n}$ is *homogeneous* if, given any $p \in X$ and any complex number c, we have $cp \in X$. The key definition is the following. An involutive homogeneous algebraic variety in $\mathbf{C}^{2n}$ is said to be *minimal* if it has no proper involutive homogeneous subvarieties. In particular, it must be irreducible, since the components of an involutive variety are also involutive. Of course an irreducible homogeneous Lagrangian variety is always minimal. The crux of the work of Bernstein and Lunts was their realization that generic hypersurfaces are also minimal. Since this result is very important in this paper, we state it with care.

Let $S_n = \mathbf{C}[x_1, \ldots, x_{2n}]$ be the polynomial ring in $2n$ variables. Denote by $S_n(k)$ the vector space of homogeneous polynomials of degree k in S_n. It has a natural structure of affine space. Thus we say that a property **P** holds *generically* in $S_n(k)$ if the set $\{f \in S_n(k) : \mathbf{P} \text{ does not hold for } f\}$ is contained in the union of a countable number of hypersurfaces of $S_n(k)$. Most often we will simply say that if f is generic then **P** holds, instead of using the carefully worded statement above.

Let $f \in S_n(k)$. The vector field $h_f = \mathcal{I}(df) = \{f, \cdot\}$ is called the *Hamiltonian field* of $\mathbf{C}^{2n}$ defined by f. An algebraic variety X in $\mathbf{C}^{2n}$ is *invariant* under h_f if its ideal $I(X)$ satisfies $h_f(I(X)) \subseteq I(X)$. The following result is equivalent to [Lun, Theorem 1].

THEOREM 2.2. *Let $n \geq 2$ and $k \geq 4$ be integers. Suppose that f is a generic homogeneous polynomial of degree k in S_n. The only algebraic varieties of $\mathbf{C}^{2n}$ of dimension greater than 1 invariant under h_f are $\mathbf{C}^{2n}$ itself, and the hypersurface $f = 0$.*

An immediate consequence of this theorem is the following corollary.

COROLLARY 2.3. *Let $n \geq 2$ and $k \geq 4$ be integers. Suppose that f is a generic homogeneous polynomial of degree k in S_n. The hypersurface $f = 0$ is a minimal involutive homogeneous subvariety of $\mathbf{C}^{2n}$.*

3. PRODUCTS

In this section we construct minimal homogeneous involutive varieties of $\mathbf{C}^{2n}$ of codimension k, for every $1 \leq k \leq n$. These varieties are obtained as products of involutive varieties. We begin with a technical lemma.

LEMMA 3.1. *Let X be an irreducible involutive homogeneous subvariety of $\mathbf{C}^{2n}$ contained in a hyperplane. Then there exists an irreducible involutive homogeneous subvariety X_0 of $\mathbf{C}^{2(n-1)}$ and $\tau \in \mathrm{Sp}(2n)$ such that*

$$\tau(X) = X_0 \times \mathbf{C} \times \{0\}.$$

If the hypersurface is $x_{2n} = 0$, then we can choose τ to be the identity map.

Proof. By Theorem 2.1 there exists $\tau \in \mathrm{Sp}(2n)$ such that $\tau(X) \subseteq \mathcal{Z}(x_{2n})$. To simplify the notation we shall assume, from now on, that X itself is contained in the hyperplane $x_{2n} = 0$.

Let $S_n = \mathbf{C}[x_1, \ldots, x_{2n}]$ and denote by S_{n-1} the subring $\mathbf{C}[x_1, \ldots, x_{2(n-1)}]$. Every element of $I(X)$ can be written in the form $g + x_{2n}h$, where $h \in S_n$ and

$$g = \sum_{i=0}^{m} a_i x_{2n-1}^i \in S_{n-1}[x_{2n-1}],$$

with $a_m \neq 0$. Since $X \subseteq \mathcal{Z}(x_{2n})$, it follows that $g \in I(X)$. But X is involutive, and $x_{2n} \in I(X)$, thus

$$\frac{\partial g}{\partial x_{2n-1}} = \{g, x_{2n}\} \in I(X).$$

Therefore, differentiating g m-times with respect to x_{2n-1}, we conclude that $a_m \in I(X)$. It follows by induction that $a_m, \ldots, a_0 \in I(X)$. Writing $I_0 = I(X) \cap S_{n-1}$, we have proved that

$$I(X) = I_0[x_{2n-1}] + S_n x_{2n}.$$

Moreover, I_0 is clearly a prime ideal of S_{n-1} closed under the Poisson bracket.

Now let X_0 be the zero set of I_0 in $\mathbf{C}^{2(n-1)}$. We have proved that X_0 is an irreducible, involutive, homogeneous, subvariety of $\mathbf{C}^{2(n-1)}$ and that

$$X = X_0 \times \mathbf{C} \times \{0\}.$$

THEOREM 3.2. *Let $n \geq 2$ and $m \geq 0$ be integers. Suppose that X is a minimal involutive homogeneous subvariety of $\mathbf{C}^{2n}$ and that L is a linear Lagrangian subvariety of $\mathbf{C}^{2m}$. Then $X \times L$ is a minimal involutive homogeneous subvariety of $\mathbf{C}^{2(m+n)}$.*

Proof. Denote by $x_1, \ldots, x_{2m}$ the coordinates of $\mathbf{C}^{2m}$. By Theorem 2.1, we can assume that L has equations $x_2 = x_4 = \cdots = x_{2m} = 0$ in $\mathbf{C}^{2m}$.

The proof is by induction on m. If $m = 0$, there is nothing to do. Suppose that $m \geq 1$. Let Y be an irreducible involutive homogeneous subvariety of $X \times L$. Since

$$Y \subseteq X \times L \subseteq \mathscr{Z}(x_{2m}),$$

it follows from Lemma 3.1 that $Y = Y_0 \times \mathbf{C} \times \{0\}$, for some irreducible involutive homogeneous subvariety Y_0 of $\mathbf{C}^{2(n+m-1)}$. Note that Y_0 is actually contained in $X \times (\mathbf{C} \times \{0\})^{m-1}$. But, by the induction hypothesis, the latter is a minimal involutive homogeneous subvariety of $\mathbf{C}^{2(n+m-1)}$. Hence it must be equal to Y_0. Thus

$$Y = (X \times (\mathbf{C} \times \{0\})^{m-1}) \times \mathbf{C} \times \{0\} = X \times L,$$

and so $X \times L$ is a minimal involutive homogeneous subvariety of $\mathbf{C}^{2(n+m)}$.

We are now ready to show that $\mathbf{C}^{2n}$ contains minimal involutive homogeneous subvarieties of every possible codimension.

COROLLARY 3.3. *Let $n \geq 2$ and $1 \leq k \leq n$ be integers. Then $\mathbf{C}^{2n}$ contains a minimal involutive homogeneous subvariety of codimension k.*

Proof. Since homogeneous irreducible Lagrangian varieties are minimal involutive homogeneous, the result holds when $k = n$. If $k = 1$, the result is a consequence of Corollary 2.3. Suppose that $1 < k < n$. Then

$$\mathbf{C}^{2n} = \mathbf{C}^{2(n-k+1)} \times \mathbf{C}^{2(k-1)}.$$

Let f be a generic homogeneous polynomial in $2(n-k+1)$ variables and degree greater than, or equal to 4. By Corollary 2.3, the hypersurface H of $\mathbf{C}^{2(n-k+1)}$ with equation $f = 0$ is a minimal involutive homogeneous variety. Thus, by Theorem 3.2,

$$H \times (\mathbf{C} \times \{0\})^{k-1} \subseteq \mathbf{C}^{2n}$$

is a minimal involutive homogeneous variety. Since this variety has codimension k, the corollary is proved.

The next theorem shows that we can replace the linear Lagrangian variety in Theorem 3.2 by any minimal involutive homogeneous variety if we strengthen the hypothesis on the other component of the product.

THEOREM 3.4. *Let $m \geq n + 2 \geq 4$ be integers. Suppose that X is a minimal involutive homogeneous subvariety of $\mathbf{C}^{2n}$ and that H is a generic homogeneous hypersurface in $\mathbf{C}^{2m}$ of degree greater than, or equal to 4. Then $X \times H$ is a minimal involutive homogeneous subvariety of $\mathbf{C}^{2(m+n)}$.*

Proof. Let $x_1, \ldots, x_{2n}$ be the coordinate functions of $\mathbf{C}^{2n}$ and $y_1, \ldots, y_{2m}$ be the coordinate functions of $\mathbf{C}^{2m}$. The corresponding polynomial rings will be denoted by $\mathbf{C}[\boldsymbol{x}]$ and $\mathbf{C}[\boldsymbol{y}]$, respectively. Let π be the projection of $\mathbf{C}^{2n} \times \mathbf{C}^{2m}$ on the first component of the product.

Suppose that W is a homogeneous involutive irreducible subvariety of $X \times H$. Since W is an involutive variety in $\mathbf{C}^{2(n+m)}$, we have that $\dim W \geq m + n$. Thus, by the theorem of the dimension of the fibres [Har, Theorem 11.12, p. 138], if $p \in W$, then

$$\dim(\pi^{-1}(\pi(p)) \geq \dim W - 2n \geq m - n \geq 2. \tag{3.5}$$

Let $\pi(p) = p_1$, and consider the ideal $J = \{g(p_1, y) : g \in I(W)\}$. Denote by V the variety of zeroes of J in $\mathbf{C}^{2m}$. We have that $\pi^{-1}(p_1) = \{p_1\} \times V$, and it follows from (3.5) that $\dim V \geq 2$. On the other hand, H is a hypersurface of $\mathbf{C}^{2m}$ with equation $f = 0$, where f is a homogeneous polynomial in the ys. Let $g \in I(W)$. Since $I(W)$ is involutive and contains f, it follows that $\{g, f\} \in I(W)$. But $f \in \mathbf{C}[y_1, \ldots y_{2m}]$, so that $\{x_i, f\} = 0$, for $1 \leq i \leq 2n$. In particular

$$\{g(p_1, y), f\} = \{g(x, y), f\}|_{x=p_1} \in J.$$

Therefore J is invariant under the Hamiltonian vector field h_f of $\mathbf{C}^{2n}$.

Let us show that the radical of J is also invariant under h_f. Suppose that $g \in \mathbf{C}[\boldsymbol{y}]$ satisfies $g^k \in J$, for some positive integer k. Since J is invariant under h_f, it follows that $h_f^k(g^k) \in J$. A simple calculation shows that $(h_f(g))^k$ belongs to the radical of J, and this implies that the radical is stable under h_f. Hence V is a subvariety of H invariant under h_f.

Since f is a homogeneous generic polynomial of degree greater than, or equal to 4, and $\dim V \geq 2$, it follows from Theorem 2.2 that $V = H$. Thus $\pi^{-1}(p_1) = \{p_1\} \times H$. Since this holds for all points $p_1 \in \pi(W)$, it follows that $W = \pi(W) \times H$.

Now let $J' = \{g(x, 0) : g \in I(W)\} \subseteq \mathbf{C}[\boldsymbol{x}]$ and let Y be the variety of zeroes of J' in $\mathbf{C}^{2n}$. We have that $\pi(W) = Y$. Hence $J' = I(W) \cap \mathbf{C}[\boldsymbol{x}]$. Since $I(W)$ is an involutive prime ideal of $\mathbf{C}[\boldsymbol{x}, \boldsymbol{y}]$, it follows that J' is an involutive prime ideal of $\mathbf{C}[\boldsymbol{x}]$. Hence Y is a homogeneous involutive subvariety of $\mathbf{C}^{2n}$ contained in X. But X is a minimal involutive homogeneous subvariety of $\mathbf{C}^{2n}$, so $X = Y$. Thus $W = X \times H$, and the proof is complete.

4. MODULES OVER THE WEYL ALGEBRA

We are now ready to apply the results of the previous section to the representation theory of the Weyl algebra. We begin with a review of some basic facts about the Weyl algebra that will help us fix the notation used throughout this section. For details see [Cou1].

The nth complex Weyl algebra A_n is generated by the coordinate functions $z_1, \ldots, z_n$ and by the corresponding partial differential operators, which will be denoted by $\partial_1, \ldots, \partial_n$. The *Bernstein filtration* of A_n is obtained by giving degree 1 to $z_1, \ldots, z_n, \partial_1, \ldots, \partial_n$. The vector space of operators of A_n of degree less than, or equal to k will be denoted by B_k.

Recall that $S_n = \mathbf{C}[x_1, \ldots, x_{2n}]$. The *symbol map* of degree k is the linear map of vector spaces $\sigma_k\colon B_k \to S_n(k)$, induced by the natural projection onto B_k/B_{k-1}. Write $x_{2i-1} = \sigma_1(z_i)$ and $x_{2i} = \sigma_1(\partial_i)$, for $1 \le i \le n$. Then S_n is isomorphic to the graded algebra of A_n associated with the Bernstein filtration. If $d \in B_k \setminus B_{k-1}$, we say that its *degree* is k, and that its *principal symbol* is $\sigma(d) = \sigma_k(d)$.

Let M be a finitely generated left A_n-module. A *good filtration* of M is a filtration $\mathcal{F}$, associated to the Bernstein filtration, such that $gr^{\mathcal{F}} M$ is finitely generated over S_n. Let $\mathrm{Ch}(M)$ denote the variety of zeroes in $\mathbf{C}^{2n}$ of the annihilator of $gr^{\mathcal{F}} M$ in S_n. It is an invariant of M, called its *characteristic variety*. Note that $\mathrm{Ch}(M)$ is a *homogeneous* variety. The *codimension* of a module is the codimension of its characteristic variety. It follows from [Gab, Theorem I'_k, p. 456] that the characteristic variety of a finitely generated A_n-module is involutive.

The full subcategory of finitely generated A_n-modules whose characteristic varieties have all irreducible components minimal of codimension k is denoted by $\mathcal{M}_n^k$. Thus $\mathcal{M}_n^n$ is the category of holonomic modules. The following result, obtained by combining [BL, Sect. 2] with Corollary 2.3, shows that $\mathcal{M}_n^1$ has uncountably many non-isomorphic irreducible objects.

THEOREM 4.1. *Let $n \ge 2$ and $k \ge 4$ be integers. Suppose that d is an operator of degree k in A_n whose principal symbol is generic in $S_n(k)$. Then $A_n/A_n d$ is an irreducible left A_n-module.*

The categories $\mathcal{M}_n^k$ share many of the properties that make holonomic modules so pleasant. For example, they are closed under submodules, quotients, and extensions; and their objects have finite length. Furthermore, they have a duality functor defined using Ext^k. For details see [BL, Sect. 3].

Let M be a left A_m-module and N a left A_n-module. The *external product* $M \widehat{\otimes} N$ is a left A_{m+n}-module. As a vector space it is isomorphic to the tensor product of M by N over $\mathbf{C}$. The left action of A_{m+n} on $M \widehat{\otimes} N$ is defined as follows. First, every element of A_{m+n} can be written as a finite sum of terms of the form ab, where $a \in A_m$ and $b \in A_n$. Thus if $u \in M$ and $v \in N$, we set $ab(u \otimes v) = au \otimes bv$. Moreover $\mathrm{Ch}(M \widehat{\otimes} N) = \mathrm{Ch}(M) \times \mathrm{Ch}(N)$. See [Cou1, Chap. 13]. Before we tackle the main theorems we need a simple lemma.

LEMMA 4.2. *Let $m, n \geq 1$ be integers. Suppose that M and N are irreducible modules over A_m and A_n, respectively. Then $M \widehat{\otimes} N$ is an irreducible A_{m+n}-module.*

Proof. We wish to show that $M \widehat{\otimes} N$ is irreducible. Let $\sum_{i=1}^{s} u_i \otimes v_i$ be a non-zero element of the external product. Without loss of generality we can assume that the vs are linearly independent over $\mathbf{C}$ and that $u_1 \neq 0$. But $\operatorname{End}_{A_n}(N) \cong \mathbf{C}$ by Quillen's lemma; see [McR, 9.7.3]. Thus, by the density theorem, there exists $d \in A_n$ such that $dv_1 \neq 0$, and $dv_i = 0$ for $2 \leq i \leq s$. Hence

$$(1 \otimes d) \sum_{i=1}^{s} (u_i \otimes v_i) = u_1 \otimes dv_1 \neq 0.$$

Since M and N are irreducible, the result follows.

THEOREM 4.3. *Let $n \geq 2$ and $1 \leq k \leq n$ be integers. The category $\mathscr{M}_n^k$ has uncountably many non-isomorphic irreducible objects of codimension k.*

Proof. This is well-known when $k = n$; while if $k = 1$, it follows from Theorem 4.1. So we may assume that $1 < k < n$. In the proof of Corollary 3.3 we showed that if H is a generic hypersurface of $\mathbf{C}^{2(n-k+1)}$ with $\deg H \geq 4$ then

$$H \times (\mathbf{C} \times \{0\})^{k-1}$$

is a minimal involutive homogeneous variety of $\mathbf{C}^{2n}$.

By Theorem 4.1 there exists an irreducible A_{n-k+1}-module M with $\operatorname{Ch}(M) = H$. Let $\mathscr{O}(\mathbf{C}^{k-1})$ be the coordinate ring of $\mathbf{C}^{k-1}$ considered as an A_{k-1}-module in the usual way. Then $\operatorname{Ch}(\mathscr{O}(\mathbf{C}^{k-1})) = (\mathbf{C} \times \{0\})^{k-1}$ in $\mathbf{C}^{2(k-1)}$. The external product $M \widehat{\otimes} \mathscr{O}(\mathbf{C}^{k-1})$ is an irreducible A_n-module by Lemma 4.2. Its characteristic variety is $H \times (\mathbf{C} \times \{0\})^{k-1}$, which is a minimal involutive variety by Theorem 3.2.

Recall that the characteristic variety is an invariant of the module. Thus if we repeat the construction above with a different H, we will obtain a non-isomorphic irreducible module. Since there are uncountably many possible choices for H, the proof of the theorem is complete.

Let us now turn to extension groups. Suppose that $1 \leq m \leq n$ are integers. Let i: $\mathbf{C}^m \to \mathbf{C}^n$ be the *standard embedding*, that is, the map given by $i(v) = (v, 0)$. Denote by W the linear subspace of $\mathbf{C}^n$ obtained by equating to zero its last $n - m$ coordinate functions. By Kashiwara's equivalence the direct image functor of $\mathscr{D}$-modules is an equivalence between the bounded derived category of finitely generated A_m-modules and the bounded derived category of finitely generated A_n-modules with support in W. For details see [Bor, Chap. VI, Theorem 7.13, p. 264]. Denoting the direct image of an A_m-module M under the map i by $i_\star(M)$, we have the following immediate consequence of Kashiwara's equivalence.

LEMMA 4.4. *Let $1 \leq m < n$ be integers and let $i\colon \mathbf{C}^m \to \mathbf{C}^n$ be the standard embedding. If M and N are finitely generated left A_m-modules, then*

$$\mathrm{Ext}^1_{A_n}(i_\star N, i_\star M) \cong \mathrm{Ext}^1_{A_m}(N, M)$$

as complex vector spaces.

Our aim is to generalize the results of [Cou4] to higher dimensions. Let $\mathbb{V}$ be the category of complex vector spaces and let

$$\mathcal{F}\colon \mathcal{M}^t_n \times \mathcal{M}^s_n \to \mathbb{V}$$

be a bifunctor. We say that $\mathcal{F}$ is *finite dimensional* if $\mathcal{F}(M, N)$ is a finite dimensional vector space for all objects M of $\mathcal{M}^t_n$ and N of $\mathcal{M}^s_n$.

THEOREM 4.5. *Let $n \geq 2$ and $1 \leq t, s \leq n$ be integers. The bifunctor*

$$\mathrm{Ext}^1(\cdot, \cdot) : \mathcal{M}^t_n \times \mathcal{M}^s_n \to \mathbb{V}$$

is finite dimensional if and only if $s = t = n$ or $t - s \geq 1$.

Proof. If $t = s = n$, then Ext^1 is finite dimensional by [Ka, Theorem 4.8, p. 135]; see also [Bjk, Theorem 6.6, p. 20]. If $t - s \geq 1$, then the functor is finite dimensional by [Cou4, Proposition 2.3]. Since there are no integral points between the lines $t = s$ and $t = s + 1$, the theorem will be proved if we show that Ext^1 is infinite dimensional if $t < s \leq n$ or $t = s < n$.

Recall that we are denoting the coordinates of $\mathbf{C}^n$ by $z_1, \ldots, z_n$, and those of $\mathbf{C}^{2n}$ by $x_1, \ldots, x_{2n}$. Moreover, ∂_i stands for the partial differential operator $\partial/\partial z_i$, so that $x_i = \sigma_1(z_i)$ and $x_{2i} = \sigma(\partial_i)$, for $1 \leq i \leq n$. Finally, if $m \leq n$, then A_m will stand for the mth Weyl algebra generated by $z_1, \ldots, z_m$ and $\partial_1, \ldots, \partial_m$.

Let $0 < k < n$ be an integer and let $d_k \in A_{n-k+1}$ be an operator of degree greater than, or equal to 4, whose principal symbol is a generic polynomial. Write $M_k = A_{n-k+1}/A_{n-k+1}d_k$. Since $n - k + 1 \geq 2$, it follows from Theorem 4.1 that M_k is an irreducible object of the category $\mathcal{M}^1_{n-k+1}$. Let $i_k\colon \mathbf{C}^{n-k+1} \to \mathbf{C}^n$ be the standard embedding and denote by $\mathbf{C}^{2(k-1)}$ the subspace of $\mathbf{C}^n$ with coordinates $x_{2(n-k+2)-1}, \ldots, x_{2n}$. Then

$$(i_k)_\star(M_k) \cong M_k \widehat{\otimes} \mathbf{C}[\partial_{n-k+2}, \ldots, \partial_n]$$

has characteristic variety $H \times L$, where L is the linear lagrangian subspace of $\mathbf{C}^{2(k-1)}$ with equations $x_{2(n-k+2)} = x_{2(n-k+3)} \cdots = x_{2n} = 0$. Thus, by Theorem 3.2, $(i_k)_\star(M_k)$ is an object of $\mathcal{M}^k_n$. Note that if $k = 1$ then i_1 is the identity map. If $k = n$, it is convenient to write $M_n = \mathbf{C}$, so that $i_\star M_n = \mathbf{C}[\partial_1, \ldots, \partial_n]$.

By Lemma 4.4 we have that

$$\mathrm{Ext}^1((i_t)_\star(M_t), (i_t)_\star(M_t)) \cong \mathrm{Ext}^1(M_t, M_t)$$

which is an infinite dimensional complex vector space for $t < n$, by [Cou2, Theorem 4.5]. This settles the case $s = t < n$. Now suppose that $t < s \leq n$. Denoting by j the standard embedding of $\mathbf{C}^{n-s+1}$ into $\mathbf{C}^{n-t+1}$, we have that $i_s = i_t \circ j$. Thus, by Lemma 4.4 and [Cou1, Theorem 18.2.1, p. 165],

$$\mathrm{Ext}^1((i_t)_\star(M_t), (i_s)_\star(M_s)) \cong \mathrm{Ext}^1(M_t, j_\star M_s).$$

Since $j_\star M_s$ is isomorphic to the quotient of A_{n-t+1} by the left ideal J generated by d_s and $\partial_{n-s+2}, \ldots, \partial_{n-t+1}$, its symbol ideal is clearly prime. Thus $\mathrm{Ext}^1(M_t, j_\star M_s)$ is an infinite dimensional complex vector space by [Cou4, Theorem 2.5].

The next result gives a homological characterization of the category of holonomic modules. This generalizes the (incorrectly stated) Corollary 3.5 of [Cou4].

COROLLARY 4.6. *Let $n \geq 2$ and $1 \leq t, s \leq n$ be integers. The bifunctor*

$$\mathrm{Ext}^1(\cdot, \cdot) : \mathscr{M}_n^t \times \mathscr{M}_n^s \to \mathbb{V}$$

is finite dimensional for all $1 \leq s \leq n$, if and only if, $t = n$.

5. HOMOGENEOUS LAGRANGIAN VARIETIES

In this section we use the techniques developed in [BL] to study the geometric properties of normal homogeneous Lagrangian subvarieties of complex $2n$-space. We begin with a technical lemma.

LEMMA 5.1. *Let X be an irreducible subvariety of $\mathbf{C}^m$ of codimension s and let $\alpha \in \Omega^1(\mathbf{C}^m)$. Suppose that $\alpha|_{T_pX} = 0$ for every $p \in X_0 = X \setminus \mathrm{Sing}(X)$. If $I(X)$ is generated by polynomials $f_1, \ldots, f_s$, then*

$$1 \otimes \alpha \in \sum_{i=1}^{s} \mathscr{O}(X_0)(1 \otimes df_i)$$

in $\mathscr{O}(X_0) \otimes_{\mathscr{O}(\mathbf{C}^m)} \Omega^1(\mathbf{C}^m)$.

Proof. Let V be an affine open set of $\mathbf{C}^m \setminus \mathrm{Sing}(X)$. Denote by $\boldsymbol{m}_p$ the maximal ideal of $\mathscr{O}(X \cap V)$ that corresponds to $p \in X \cap V$ under the Nullstellensatz. Since $X \cap V$ is a smooth subvariety of V, the sequence

$$0 \to \frac{I(X \cap V)}{I(X \cap V)^2} \xrightarrow{\psi} \mathscr{O}(X \cap V) \otimes_{\mathscr{O}(V)} \Omega^1(V) \xrightarrow{\phi} \Omega^1(X \cap V) \to 0 \tag{5.2}$$

is split exact. Thus it will remain exact if it is tensored with $\mathscr{O}(X \cap V)/\boldsymbol{m}_p$, for some $p \in X \cap V$.

Since $\alpha|_{T_pX} = 0$, we have that

$$\phi(1 \otimes \alpha) \equiv 0 \pmod{\boldsymbol{m}_p \Omega^1(X \cap V)}$$

for every $p \in X \cap V$. However, $\Omega^1(X \cap V)$ is a projective $\mathscr{O}(X \cap V)$-module, and $\mathscr{O}(X \cap V)$ is a Jacobson ring; so $\phi(1 \otimes \alpha) = 0$. Hence $1 \otimes \alpha \in \operatorname{Im}(\psi)$. Thus there exist $g_1, \ldots, g_s \in \mathscr{O}(V \cap X)$ such that $1 \otimes \alpha = \sum_{i=1}^{s} g_i(1 \otimes df_i)$.

Let V' be another affine open set of $\mathbf{C}^m \setminus \operatorname{Sing}(X)$, and write $1 \otimes \alpha \in \sum_{i=1}^{s} g_i'(1 \otimes df_i)$, for $g_1', \ldots, g_s' \in \mathscr{O}(X \cap V')$. Since $I(X)$ is generated by $f_1, \ldots, f_s$ and X has codimension s, there exists an open affine set $U \subseteq V \cap V'$ such that the $\mathscr{O}(X \cap U)$-module $I(X \cap U)/I(X \cap U)^2$ is free on the classes of $f_1, \ldots, f_s$ in this quotient; see [Ku, Corollary V.5.11, p. 153]. Thus $\operatorname{Im}(\psi|_U)$ is free on $1 \otimes df_1, \ldots, 1 \otimes df_s$. Hence $g_i|_U = g_i'|_U$. It follows that the g_i can be extended to the whole of $\mathbf{C}^m \setminus \operatorname{Sing}(X)$, and the lemma is proved.

Recall from Sec. 2 that there exists an isomorphism $\mathscr{J}\colon T^\star(\mathbb{C}^{2n}) \to T\mathbb{C}^{2n}$. Let E_n be the Euler vector field of $\mathbf{C}^{2n}$, and let $\alpha_n = -\mathscr{J}^{-1}(E_n)$ be the corresponding 1-form; thus

$$\alpha_n = \sum_{i=1}^{n} x_{2i-1}\, dx_{2i} - x_{2i}\, dx_{2i-1}.$$

Let X be an irreducible, proper, homogeneous, subvariety of $\mathbf{C}^{2n}$. We say that X is *amenable* if there exist homogeneous polynomials $f_1, \ldots, f_s$ in $I(X)$ such that

$$\alpha_n \in \sum_{i=1}^{s} S_n\, df_i + I(X)\Omega^1(\mathbf{C}^{2n}).$$

Note that, without loss of generality, we may assume that $f_1, \ldots, f_s$ generate $I(X)$.

LEMMA 5.3. *Let $n \geq 1$ be an integer and let X be an irreducible, proper, homogeneous, involutive subvariety of $\mathbf{C}^{2n}$. If X is amenable, then there exists $\tau \in \operatorname{Sp}(2n)$ such that*

$$\tau(X) = \mathscr{Z}(x_2, \ldots, x_{2n}) \cong \mathbf{C}^n.$$

Proof. The proof is by induction on n. If $n = 1$, then X is a homogeneous hypersurface of $\mathbf{C}^2$. Since X is irreducible, it must be a line through the origin. By Theorem 2.1 there exists $\tau \in \operatorname{Sp}(2n)$ such that $\tau(X) = \mathscr{Z}(x_2)$, as required. Suppose now that X is an irreducible, proper, involutive, *amenable* subvariety of $\mathbf{C}^{2n}$, for some $n \geq 2$.

By the amenability condition there exist homogeneous generators $f_1, \ldots, f_s$ of $I(X)$, 1-forms $\eta_i \in \Omega^1(\mathbf{C}^{2n})$ and polynomials $g_i \in S_n$, such that

$$\alpha_n = \sum_{i=1}^{s} g_i\, df_i + \sum_{i=1}^{s} f_i \eta_i.$$

But $\Omega^1(\mathbf{C}^{2n})$ is a graded S_n-module. Since α_n has degree 1, we can assume that only the homogeneous components of degree 1 of each term in the sum above have been retained.

Suppose, by contradiction, that $\deg f_i \geq 2$, for $1 \leq i \leq s$. Then $\sum_{i=1}^{s} f_i \eta_i = 0$, for if it were non-zero, then it would have degree at least 2. Moreover, $g_i \neq 0$ can occur only if the corresponding f_i has degree exactly 2. Now, since $\alpha_n \neq 0$, at least one of the f_i must have degree 2. Renumber the fs so that $f_1, \ldots, f_t$ have degree 2 for some $t \leq s$. Note that $g_1, \ldots, g_t$ must then be constants. Thus, if $Q = \sum_{i=1}^{t} g_i f_i$, then $\alpha_n = dQ$. But $d\alpha_n = 2\omega_n$, so α_n is not a closed 1-form, and we have a contradiction. Therefore at least one of the fs must have degree 1. Hence X is contained in a hyperplane, and by Lemma 3.1 there exists an irreducible homogeneous Lagrangian subvariety X_0 of $\mathbf{C}^{2(n-1)}$ and $\theta_1 \in \mathrm{Sp}(2n)$ such that

$$\theta_1(X) = X_0 \times \mathbf{C} \times \{0\}.$$

In order to use the induction hypothesis, we must show that X_0 is amenable.

Since $I(X) = I(X_0)[x_{2n-1}] + S_n x_{2n}$, there exists $h_i \in I(X_0)$ such that $f_i - h_i \in S_n x_{2n-1} + S_n x_{2n}$. Thus

$$df_i - dh_i \in \sum_{j=2n-1}^{2n} \left(x_j \Omega^1(\mathbf{C}^{2n}) + S_n\, dx_j\right),$$

and so

$$\alpha_n \in \sum_{i=1}^{s} S_{n-1}\, dh_i + I(X_0)\Omega^1(\mathbf{C}^{2(n-1)}) + \sum_{j=2n-1}^{2n} \left(x_j \Omega^1(\mathbf{C}^{2n}) + S_n\, dx_j\right).$$

But $\Omega^1(\mathbf{C}^{2n})$ is free with basis $dx_1, \ldots, dx_{2n}$, so that

$$\alpha_{n-1} \in \sum_{i=1}^{s} S_{n-1}\, dh_i + I(X_0)\Omega^1(\mathbf{C}^{2(n-1)}).$$

Thus X_0 is amenable.

By the induction hypothesis there exists $\theta_2 \in \mathrm{Sp}(2(n-1))$ such that $\theta_2(X_0)$ is the variety $x_2 = \cdots = x_{2(n-1)} = 0$ of $\mathbf{C}^{2(n-1)}$. Extending θ_2 to $\mathrm{Sp}(2n)$ so that it is the identity on x_{2n-1} and x_{2n}, and setting $\tau = \theta_2\theta_1$, we conclude that $\tau(X) = \mathcal{Z}(x_2, \ldots, x_{2n})$, as required.

THEOREM 5.4. *Let $n \geq 1$ and let X be an irreducible homogeneous Lagrangian subvariety of $\mathbf{C}^{2n}$. If X is a global complete intersection and its singular locus has codimension at least 2, then there exists $\tau \in \mathrm{Sp}(2n)$ such that*

$$\tau(X) = \mathscr{Z}(x_2, \dots, x_{2n}) \cong \mathbf{C}^n.$$

Proof. We first prove that X is amenable. Let $f_1, \dots, f_n$ be homogeneous polynomials that generate $I(X)$. Put $X_0 = X \setminus \mathrm{Sing}(X)$. If $p \in X_0$, then

$$T_pX = \{u \in \mathbf{C}^{2n} : df_i(p) \cdot u = 0, \ \text{for } 1 \leq i \leq n\}.$$

But X is involutive, so the Hamiltonian vector fields of the fs at p generate a subspace of T_pX. Since the isomorphism $\mathscr{I}$ of Sec. 2 maps $df_i(p)$ to $h_{f_i}(p)$, this subspace must have dimension n. So T_pX is generated by the Hamiltonian vector fields $h_{f_i}(p)$, for $1 \leq i \leq n$. Now

$$\alpha_n(h_{f_i}) = \omega_n(E_n, h_{f_i}) = E_n(f_i) = (\deg(f_i))f_i.$$

In particular, α_n restricted to T_pX is zero, for every $p \in X_0$.

It follows from Lemma 5.1 that there exist $g_1, \dots, g_n \in \mathscr{O}(X_0)$ such that

$$1 \otimes \alpha_n = \sum_{i=1}^{n} g_i(1 \otimes df_i) \in \mathscr{O}(X_0) \otimes \Omega^1(\mathbf{C}^{2n}).$$

Since X is a global complete intersection that is regular in codimension 1, it follows from [Hart, Proposition II.8.23, p. 186] that X is normal. Thus the gs can be extended to the whole of X by [Iit, Theorem 2.15, p. 124]. Using bars to denote classes modulo $I(X)$, we have that

$$\overline{\alpha_n} = \sum_{i=1}^{s} g_i \, \overline{df_i} \in \frac{\Omega^1(\mathbf{C}^{2n})}{I(X)\Omega^1(\mathbf{C}^{2n})} \cong \mathscr{O}(X) \otimes \Omega^1(\mathbf{C}^{2n}).$$

Denoting the lifting of g_i to $S_n = \mathbf{C}[x_1, \dots, x_{2n}]$ also by g_i, we conclude that

$$\alpha_n \in \sum_{i=1}^{s} g_i \, df_i + I(X)\Omega^1(\mathbf{C}^{2n}).$$

which is the condition we had to prove. The theorem now follows immediately from Lemma 5.3.

A homogeneous variety in $\mathbf{C}^k$ gives rise to a projective variety in $\mathbf{P}^{k-1}(\mathbf{C})$, called its *projectivization*. The following corollary generalizes the first part of the proof of [BL, Theorem 1, p. 229].

COROLLARY 5.5. *Let $n \geq 1$ and let X be an irreducible homogeneous Lagrangian subvariety of $\mathbf{C}^{2n}$. If X is a global complete intersection whose projectivization is normal, then there exists $\tau \in \mathrm{Sp}(2n)$ such that*

$$\tau(X) = \mathscr{Z}(x_2, \dots, x_{2n}) \cong \mathbf{C}^n.$$

Proof. Since the projectivization of X is normal, it is regular in codimension 1. Thus X itself is regular in codimension 1. The result now follows by Theorem 5.4.

Examples of homogeneous Lagrangian varieties are easily constructed. Let $n \geq 2$, and let $f \in \mathbf{C}[x_1, x_3, \dots, x_{2n-1}]$ be a homogeneous polynomial. Denote by X the linear subspace of $\mathbf{C}^{2n}$ with equations $x_2 = x_4 = \cdots = x_{2n} = 0$, and identify $\mathbf{C}^{2n}$ with the cotangent bundle T^*X. Let Y be the hypersurface of X with equation $f = 0$.

Suppose that Y has only an isolated singularity at the origin. Equivalently, $\nabla f(p) \neq 0$, if $p \neq 0$. Thus $Y_0 = Y \setminus \{0\}$ is a smooth subvariety of $X_0 = X \setminus \{0\}$. Let $C(f)$ be the closure of the conormal bundle $T^*_{Y_0} X_0$ in the Zariski topology of $\mathbf{C}^{2n}$. It is easy to check that $C(f)$ is an irreducible homogeneous Lagrangian variety of $\mathbf{C}^{2n}$. Moreover, $C(f)$ is smooth outside the zero fiber $T^*_0 X$.

Now every homogeneous global complete intersection whose singular locus has codimension greater than 2 is normal. Thus it seems reasonable to ask if Theorem 5.4 will hold if X is a global complete intersection is replaced by the weaker condition X is normal. We give an example to show that the answer is no. Let $q_n = x_1^2 + x_3^2 + \cdots + x_{2n-1}^2$. It is shown in [CouL, Lemma 5.1, p. 193] that, in this case, $C(q_n)$ has equations

$$\begin{aligned} x_1^2 + x_3^2 + \cdots + x_{2n-1}^2 &= 0, \\ x_2^2 + x_4^2 + \cdots + x_{2n}^2 &= 0, \\ x_1 x_2 + \cdots + x_{2n-1} x_{2n} &= 0, \\ x_{2i-1} x_{2j} - x_{2i} x_{2j-1} &= 0, \end{aligned}$$

for $1 \leq i < j \leq n$. Let $\mathscr{F}$ be the linear automorphism of $\mathbf{C}^{2n}$ defined by $\mathscr{F}(x_{2i-1}) = x_{2i}$ and $\mathscr{F}(x_{2i}) = -x_{2i-1}$, for $1 \leq i \leq n$. Clearly $\mathscr{F}(C(q_n)) = C(q_n)$, while $\mathscr{F}(T^*_0 X) = T^*_X X$, the zero section of $T^*X = \mathbf{C}^{2n}$. Hence $\mathscr{F}$ maps $C(q_n) \setminus T^*_0 X$ bijectively onto $C(q_n) \setminus T^*_X X$. Since the former is a smooth variety, so is the latter. In particular $C(q_n)$ is smooth outside the origin. With the help of the computer algebra system Macaulay one can compute a free resolution of the coordinate ring of $C(q_3)$, and show that it has depth 3. Thus $C(q_3)$ is Cohen–Macaulay. By [Mat, Theorem 23.8, p. 183] it follows that $C(q_3)$ is normal. However, $C(q_3)$ is not a linear subspace.

6. FINAL COMMENTS

The results of this paper suggest the following three problems.

Problem 6.1. Is the product of minimal involutive homogeneous varieties always a minimal involutive homogeneous variety?

Problem 6.2. Find necessary and sufficient conditions for an involutive homogeneous variety in $\mathbf{C}^{2n}$ to be a characteristic variety of a finitely generated A_n-module.

No example of a homogeneous involutive variety that is not a characteristic variety seems to be known. However, if *homogeneous* is replaced by *conic*, and the *Bernstein* filtration by the *order* filtration, then such examples exist; see [CouL, Sect. 5] and [CHL]. A finitely generated A_n-module is called *singular* if the irreducible components of its characteristic variety are minimal homogeneous involutive varieties. Note that the characteristic variety of a singular module need not be equidimensional.

Problem 6.3. Let M be a singular A_n-module. Is it true that, if the functor

$$\operatorname{Ext}^1(M, \cdot)\colon \mathscr{M}_n^k \to \mathbb{V}$$

is finite dimensional for all integers $1 \leq k \leq n$, then M is holonomic?

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